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The number of functionally independent invariants of a pseudo-Riemannian metric

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Abstract. The number of functionally independent scalar invariants of arbitrary order of a generic pseudo-Riemannian metric on an n-dimensional manifold is determined.

1. *Introduction*

The goal of this work is to determine the number $i_{n,r}$ of functionally independent differential invariants of order r of a *generic* pseudo-Riemannian metric g on an n-dimensional manifold N. The results are: for every $n \geqslant 1$, $i_{n,0} = i_{n,1} = 0$; for every $r \geqslant 0$, $i_{1,r} = 0$; $i_{2,2} = 1$; and for every $r \geqslant 3$, $i_{2,r} = \frac{1}{2}(r+1)(r-2)$; and finally,

$$i_{n,r} = n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)}\binom{n+r}{r} \qquad \text{for every } n \geqslant 3, r \geqslant 2.$$

The theory of metric invariants is classic in both general relativity (GR) and Riemannian geometry. The standard approach to this topic relies on the definition of an invariant as a polynomial in the g_{ij}'s, their partial derivatives up to a certain order, say $\partial^{|\alpha|}g_{ij}/\partial x^\alpha$, $|\alpha| \leqslant r$ and on $[\det(g_{ij})]^{-1}$, which is 'natural' under diffeomorphisms (for example, see [1]). For scalar invariants, the above definition does not allow one to pose the question of how many functionally independent invariants there are for each order since some of the functional relationships among invariants may be outside the ring prescribed by the above definition and furthermore, standard analysis tools (such as involutiveness, Frobenius theorem, etc) cannot be applied since the ring is not complete. From this point of view, the enumeration of the scalars constructed from the Riemann tensor of the Levi–Cività connection of a pseudo-Riemannian metric by means of covariant differentiation, tensor products and contractions has been discussed in some recent papers: in [2], the number of independent homogeneous scalar monomials of each order and degree up to order 12 in the derivatives of the metric is determined and in [3], the same number is determined up to order 14. Apart from the interest and complexity of these results, especially in relation to the so-called Weyl invariants (cf [4]) for field theory, it is clear that the determination of the number $i_{n,r}$ is the most relevant fact since it provides the number of essentially different Diff(N)-invariant Lagrangians of arbitrary order that exist in GR. It, thus, seems natural to base the theory on the jet-bundle notion of an invariant (cf [5]) which avoids the aforementioned difficulties of the polynomial notion and translates the naturality condition

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into an authentic condition of invariance under the action of the group of diffeomorphisms of N on an appropriate jet bundle.

The plan of this paper is as follows. In section 2, we introduce the notion of a metric invariant as well as that of an invariant Lagrangian density, although, for an oriented ground manifold N, the latter is reduced to the former since the bundle of metrics over N is endowed with a canonical invariant zero-order Lagrangian density, so that the emphasis is put on scalar invariants. The notion of invariance is related to a specific representation of the vector fields of N into vector fields of the r-jet bundle of metrics. Section 3 contains the explicit determination of this representation and its formulae are used throughout the paper. In section 4, we prove that, on a dense open subset of the r-jet bundle, the metric invariants coincide with the ring of first integrals of an involutive distribution which is obtained by linearizing the basic representation by means of a homomorphism of vector bundles Φ^r. The number of invariants $i_{n,r}$, is, thus, equivalent to knowing the rank of Φ^r. Sections 5, 6 and 7 are devoted to this aim of distinguishing the different cases that appear according to the values of order r of the jet bundle that we are considering and the dimension n of the ground manifold. Finally, section 8 contains the calculation of $i_{n,r}$ and the comparison of $i_{n,2}$ with the standard procedure (cf [6]) in order to generate the second-order metric invariants.

2. The notion of a metric invariant

Let N be an n-dimensional differentiable manifold. Given an integer $0 \leqslant n^+ \leqslant n$, we shall denote by $p : \mathcal{M} = \mathcal{M}_{n^+}(N) \to N$ the bundle of pseudo-Riemannian metrics on N of signature (n^+, n^-), $n^- = n - n^+$ (i.e. the global sections of p are exactly the pseudo-Riemannian metrics on N of signature (n^+, n^-) at each point). Let $p_r : J^r(\mathcal{M}) \to N$ be the r-jet bundle of local sections of p. The r-jet at a point $x \in N$ of a metric g of $\mathcal{M}$ will be denoted by $j^r_x(g)$. For every $r \geqslant s$, we also have a natural projection $p_{rs} : J^r(\mathcal{M}) \to J^s(\mathcal{M})$, $p_{rs}(j^r_x g) = j^s_x g$. Let $(U; x_1, \ldots, x_n)$ be an open coordinate domain of N and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index of non-negative integers. We set $|\alpha| = \sum_i \alpha_i$. The family of functions $(x_i \circ p_r, y^{jk}_\alpha)$, $j \leqslant k, |\alpha| \leqslant r$, defined by $y^{jk}_\alpha(j^r_x g) = (\partial^{|\alpha|} g_{jk}/\partial x^\alpha)(x)$, where $g_{jk} = g(\partial/\partial x_j, \partial/\partial x_k)$, constitutes a coordinate chart on $p_r^{-1}U = J^r(p^{-1}U)$. We shall simply write y_{jk} instead of y^{jk}_0. Note that the functions $(x_i \circ p, y_{jk})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k \leqslant n$ are a coordinate system on $p^{-1}(U)$. We shall also set $y^{jk}_\alpha = y^{kj}_\alpha$ for $j > k$.

Let $f: N \to N'$ be a diffeomorphism. We shall denote by $\overline{f}: \mathcal{M} \to \mathcal{M}'$, $\mathcal{M}' = \mathcal{M}_{n^+}(N')$ the natural lift of f to the bundles of pseudo-Riemannian metrics; i.e. $\overline{f}(g_x) = (f^{-1})^* g_x$. Hence, $p' \circ \overline{f} = f \circ p$.

The diffeomorphism $\overline{f}: \mathcal{M} \to \mathcal{M}'$ has a natural extension to jet bundles $J^r(f): J^r(\mathcal{M}) \to J^r(\mathcal{M}')$, defined as follows: $J^r(f)\left(j^r_x g\right) = j^r_{f(x)}(\overline{f} \circ g \circ f^{-1})$.

Given a vector field $X \in \mathfrak{X}(N)$, we shall denote by $\overline{X}^r$ the natural lift of X to $J^r(\mathcal{M})$. For $r = 0$, we shall simply write $\overline{X}$ instead of $\overline{X}^0$. Note that $\overline{X}$ is the natural lift to the bundle $\mathcal{M}$ of pseudo-Riemannian metrics of the vector field X. If φ_t is the local flow of X, then $J^r(\varphi_t)$ is the local flow of $\overline{X}^r$. Hence, $\overline{X}^r$ is projectable onto $\overline{X}^{r-1}$ and $\overline{X}$ is projectable onto X. The mapping $X \mapsto \overline{X}^r$ is an $\mathbb{R}$-linear injection and, for every $X, Y \in \mathfrak{X}(N)$,

$$[\overline{X}^r, \overline{Y}^r] = \overline{[X, Y]}^r. \tag{1}$$

Hence, we have a faithful representation of $\mathfrak{X}(N)$ into $\mathfrak{X}(J^r(\mathcal{M}))$.

Definition 1. A function $F \in C^\infty(J^r(\mathcal{M}))$ (which may only be defined on an open subset) is said to be a *metric differential invariant* of order r if, for every $X \in \mathfrak{X}(N)$, $\overline{X}^r F = 0$.

Definition 2. A function $F \in C^\infty(J^r(\mathcal{M}))$ is said to be a *metric invariant* of order r if, for every diffeomorphism $f : N \to N$, $F \circ J^r(f) = F$.

Remark 3. Metric invariants are a subring of the ring of metric differential invariants. In fact, the set of vector fields on N with compact support $\mathfrak{X}_c(N)$ is a dense ideal of $\mathfrak{X}(N)$ with respect to the C^∞ topology and, hence, a function $F \in C^\infty(J^r(\mathcal{M}))$ is a metric differential invariant if, and only if, for every $X \in \mathfrak{X}_c(N)$, $\overline{X}^r F = 0$. This is equivalent to saying that for every $t \in \mathbb{R}$, one has $F \circ J^r(\phi_t) = F$, ϕ_t being the one-parameter group generated by X with the last equation evidently holding for a metric invariant.

Example 4. Let ∇ be the Levi–Cività connection of a pseudo-Riemannian metric g of $\mathcal{M}$, and R the curvature tensor. Since R is of type $(1,3)$, for every $r \in \mathbb{N}$, $\nabla^{2r}R$ is a tensor field of type $(1, 2r+3)$. Let us choose a sequence of $r+1$ covariant indices $1 \leqslant i_0 < \ldots < i_r \leqslant 2r+3$, and let us apply to them the isomorphism $g^\sharp : T_x^*(N) \to T_x(N)$, thus obtaining a tensor field ${}^{g^\sharp}(\nabla^{2r}R)^{i_0,\ldots,i_r}$ of type $(r+2, r+2)$. If we further choose a permutation $j_1, \ldots, j_{r+2}$ of its covariant indices, we can then obtain a scalar by simply setting $S_g = c_{j_1}^1 \cdots c_{j_{r+2}}^{r+2}({}^{g^\sharp}(\nabla^{2r}R)^{i_0,\ldots,i_r})$, where c_j^i stands for the contraction of the ith contravariant index with the jth covariant index. The value of S_g at a point $x \in N$ only depends on $j_x^{2r+2}(g)$, since the local coefficients $\Gamma_{jk}^i(x)$ of ∇ only depend on $j_x^1(g)$, and R_x only depends on $j_x^2(g)$ (cf [7], IV.2.4 and III.7.6). Accordingly, we can define a function $F \in C^\infty(J^{2r+2}(\mathcal{M}))$ by imposing that $F(j_x^{2r+2}g) = S_g(x)$, and that this function is an invariant. In fact, if $f : N \to N$ is a diffeomorphism and we set $\bar{g} = (f^{-1})^*g = \overline{f} \circ g \circ f^{-1}$, then the Levi–Cività connection of $\bar{g}$ is the linear connection $\overline{\nabla}$ given by $\overline{\nabla}_X Y = f \cdot (\nabla_{f^{-1}\cdot X} f^{-1} \cdot Y)$, as follows from Koszul's formula ([7], IV.2.3), and, consequently, for every $r \in \mathbb{N}$, the tensor fields $\nabla^{2r}R$ and $\overline{\nabla}^{2r}\bar{R}$ are f-related (cf [7], VI.1.2); i.e. for every system of vector fields $X_1, \ldots, X_{2r+1} \in \mathfrak{X}(N)$, and every point $x \in N$, $f_*(\nabla^{2r}R)((X_1)_x, \ldots, (X_{2r+3})_x) = (\overline{\nabla}^{2r}\bar{R})((f \cdot X_1)_{f(x)}, \ldots, (f \cdot X_{2r+3})_{f(x)})$, or else $(\nabla^{2r}R)(X_1, \ldots, X_{2r+3}) = f^{-1} \cdot (\overline{\nabla}^{2r}\bar{R})(f \cdot X_1, \ldots, f \cdot X_{2r+3})$. Hence, $S_g(x) = S_{\bar{g}}(f(x))$, and this means $F(j_x^{2r+2}g) = F(J^{2r+2}(f)(j_x^{2r+2}g))$.

Definition 5. An rth-order *Lagrangian density* is a horizontal n-form Ω_n on $J^r(\mathcal{M})$. An rth-order Lagrangian density is said to be *invariant* if, for every $X \in \mathfrak{X}(N)$, $L_{\overline{X}^r}\Omega_n = 0$.

Remark 6. As Ω_n is horizontal, there exists locally a function $\mathcal{L} \in C^\infty(J^r\mathcal{M})$, such that $\Omega_n = \mathcal{L}\,\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. Below, we shall see that by introducing the factor $\sqrt{(-1)^{n^-}\det(y_{ij})}$ in $\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ we obtain a globally defined invariant Lagrangian density, thus reducing the problem of determining the invariant Lagrangian densities to that of the scalar invariants. Note that in this case, $\mathcal{L}$ should be substituted by $F = \mathcal{L}/\sqrt{(-1)^{n^-}\det(y_{ij})}$.

Proposition 7. Assume N is oriented. Then, the bundle of metrics $\mathcal{M}$ is endowed with a canonical invariant zero-order Lagrangian density ω_n, uniquely defined by the following condition: if $X_1, \ldots, X_n$ is an orthonormal basis for a metric g of $\mathcal{M}$, defined on an open subset $U \subset \mathcal{M}$, which belongs to the orientation of N, then for every $x \in U$, $(\omega_n)_{g_x}(\overline{X}_1, \ldots, \overline{X}_n) = 1$. Accordingly, every Lagrangian density Ω_n on $J^r(\mathcal{M})$ can be uniquely written as $\Omega_n = F\omega_n$, $F \in C^\infty(J^r(\mathcal{M}))$, and Ω_n is invariant if, and only if, F is a metric differential invariant.

Proof. Since ω_n must be a horizontal n-form, it is clear that the condition in the statement uniquely determines the desired form. Moreover, we can define a horizontal n-form on $\mathcal{M}$ by setting for every $Y_1, \ldots, Y_n \in T_{g_x}(\mathcal{M})$, $(\omega_n)_{g_x}(Y_1, \ldots, Y_n) = v_{g_x}(p_*Y_1, \ldots, p_*Y_n)$, where v_{g_x} is the Riemannian volume associated with g_x. Since $\overline{X}_i$ is p-projectable onto X_i, we have $(\omega_n)_{g_x}(\overline{X}_1, \ldots, \overline{X}_n) = v_{g_x}(X_1, \ldots, X_n)$, thus proving that ω_n satisfies the above condition.

A basis $X_1, \ldots, X_n$ for $T_x(N)$ is said to be orthonormal for the metric g_x if: $g(X_i, X_j) = \delta_{ij}$ for either $1 \leqslant i \leqslant n^+$ or $1 \leqslant j \leqslant n^+$; $g(X_i, X_j) = -\delta_{ij}$ for $n^+ + 1 \leqslant i, j \leqslant n^+ + n^-$; in other words, the matrix of g_x must be

$$\begin{pmatrix} I_{n^+} & 0 \\ 0 & -I_{n^-} \end{pmatrix}.$$

Hence, locally we have $\omega_n = \sqrt{(-1)^{n^-} \det(y_{ij})}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. Also note that ω_n cannot be considered as the volume element associated with the canonical metric $G = \sum_{i \leqslant j} y_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j$ on $\mathcal{M}$, since G is singular!

We shall now prove that ω_n is invariant. Given a diffeomorphism f of N, with the above notations, we have: $(\overline{f}^* \omega)_{g_x}(Y_1, \ldots, Y_n) = (\omega_n)_{\overline{f}(g_x)}(\overline{f}_* Y_1, \ldots, \overline{f}_* Y_n) = v_{\overline{f}(g_x)}(p_* \overline{f}_* Y_1, \ldots, p_* \overline{f}_* Y_n)$, and since $p_* \circ \overline{f}_* = f_* \circ p_*$, we obtain

$$\begin{aligned}
(\overline{f}^* \omega_n)_{g_x}(Y_1, \ldots, Y_n) &= (f^* v_{\overline{f}(g_x)})(p_* Y_1, \ldots, p_* Y_n) \\
&= (f^* v_{(f^{-1})^*(g_x)})(p_* Y_1, \ldots, p_* Y_n) \\
&= f^* (f^{-1})^* v_{g_x}(p_* Y_1, \ldots, p_* Y_n) \\
&= (\omega_n)_{g_x}(Y_1, \ldots, Y_n).
\end{aligned}$$

□

3. Local expression of the basic representation

Proposition 8. Let $X = \sum_i u_i (\partial/\partial x_i)$, $u_i \in C^\infty(U)$, $1 \leqslant i \leqslant n$, be the local expression of a vector field $X \in \mathfrak{X}(N)$ on an open coordinate domain $(U; x_1, \ldots, x_n)$ of N. The local expression of the lifting of X to the bundle of pseudo-Riemannian metrics $\overline{X} \in \mathfrak{X}(\mathcal{M})$ in the induced coordinate system $(p^{-1}(U); x_i, y_{jk})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k \leqslant n$, is given by

$$\overline{X} = \sum_i u_i \frac{\partial}{\partial x_i} + \sum_{i \leqslant j} v_{ij} \frac{\partial}{\partial y_{ij}} \qquad v_{ij} = -\sum_h \frac{\partial u_h}{\partial x_i} y_{hj} - \sum_h \frac{\partial u_h}{\partial x_j} y_{ih}. \tag{2}$$

Proof. First, note that v_{ij} is symmetric with respect to the indices i, j, so we shall also write $v_{ij} = v_{ji}$ for $i > j$. As is well known, the lift $\overline{X}$ is the unique vector field on $\mathcal{M}$ which is p-projectable onto X and leaves the 'canonical metric' $G = \sum_{i \leqslant j} y_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j$ on the manifold $\mathcal{M}$, invariant; i.e. $\overline{X} = \sum_i u_i (\partial/\partial x_i) + \sum_{i \leqslant j} v_{ij} (\partial/\partial y_{ij})$, for some functions $v_{ij} \in C^\infty(p^{-1}U)$, and $L_{\overline{X}} G = 0$. Hence,

$$L_{\overline{X}'} G = \sum_{i \leqslant j} [v_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j + y_{ij}\, \mathrm{d}u_i \otimes \mathrm{d}x_j + y_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}u_j] = 0$$

and this equation completely determines the unknown functions. □

From the general formulae for the prolongation of vector fields by infinitesimal contact transformations (e.g. see [8]), we then obtain the local expression for $\overline{X}^r$; more precisely,

$$\overline{X}^r = \sum_i u_i \frac{\partial}{\partial x_i} - \sum_h \sum_{i \leqslant j} \sum_{|\alpha|=0}^{r} \left\{ \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left[\frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+(i)}} y^{hj}_{\alpha-\beta} + \frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+(j)}} y^{ih}_{\alpha-\beta} \right] + \sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{\partial^{|\beta|} u_h}{\partial x^{\beta}} y^{ij}_{\alpha-\beta+(h)} \right\} \frac{\partial}{\partial y^{ij}_{\alpha}} \tag{3}$$

where (i) stands for the multi-index $(i) = (0, \ldots, \overset{i}{1}, \ldots, 0)$. The above equations can be obtained by either imposing that: (i) $\overline{X}^r$ is p_r-projectable onto X; and (ii) $\overline{X}^r$ leaves the generalized contact differential system $\mathcal{C}$ spanned by the one-forms on $J^r(\mathcal{M})$, $\theta^{ij}_\alpha = \mathrm{d}y^{ij}_\alpha - \sum_k y^{ij}_{\alpha+(k)} \mathrm{d}x_k$, $i, j = 1, \ldots, n$, $|\alpha| < r$, invariant; i.e. $L_{\overline{X}^r}\mathcal{C} \subset \mathcal{C}$, or by simply calculating the infinitesimal generator associated with $J^r(\phi_t)$, ϕ_t being the local flow of X.

Example 9. For $r = 1$, the above formula reads as follows

$$\overline{X}^1 = \sum_i u_i \frac{\partial}{\partial x_i} - \sum_h \sum_{i\leqslant j} \left(\frac{\partial u_h}{\partial x_i} y_{hj} + \frac{\partial u_h}{\partial x_j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} - \sum_{h,k} \sum_{i \leqslant j} \left(\frac{\partial^2 u_h}{\partial x_i \partial x_k} y_{hj} + \frac{\partial^2 u_h}{\partial x_j \partial x_k} y_{ih} + \frac{\partial u_h}{\partial x_i} y^{hj}_k + \frac{\partial u_h}{\partial x_j} y^{ih}_k + \frac{\partial u_h}{\partial x_k} y^{ij}_h \right) \frac{\partial}{\partial y^{ij}_k}. \tag{4}$$

4. The fundamental distribution

Theorem 10. With the above hypotheses and notations, we have

(i) $\overline{X}^r_{j^r_x g}$ only depends on $j^{r+1}_x(X)$.

(ii) There exists a unique homomorphism of vector bundles over $J^r(\mathcal{M})$

$$\Phi^r : p_r^* J^{r+1}(TN) \longrightarrow T(J^r\mathcal{M})$$

such that for every $X \in \mathfrak{X}(N)$, $j^r_x g \in J^r(\mathcal{M})$

$$\Phi^r(j^r_x g, j^{r+1}_x X) = \overline{X}^r_{j^r_x g}.$$

(iii) On a dense open subset $\mathcal{O}^r \subset J^r(\mathcal{M})$, the image of Φ^r defines an involutive distribution $\mathfrak{D}^r$ such that for every $j^r_x g \in \mathcal{O}^r$, $X \in \mathfrak{X}(N)$,

$$\mathfrak{D}^r_{j^r_x g} = \{\overline{X}^r_{j^r_x g}; X \in \mathfrak{X}(N)\} \subset T_{j^r_x g}(J^r\mathcal{M}).$$

(iv) A function $F \in C^\infty(\mathcal{O}^r)$ is a metric differential invariant if, and only if, F is a first integral of $\mathfrak{D}^r$.

Proof. Evaluating $\overline{X}^r$ at the point $j_x^r g$ from formula (3), we obtain

$$\overline{X}^r_{j_x^r g}(y^{ij}_\alpha) = -\sum_h \left\{ \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left[\frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+(i)}}(x) \frac{\partial^{|\alpha-\beta|} g_{hj}}{\partial x^{\alpha-\beta}}(x) + \frac{\partial^{|\beta|+1} u_h}{\partial x^{\beta+(j)}}(x) \frac{\partial^{|\alpha-\beta|} g_{ih}}{\partial x^{\alpha-\beta}}(x) \right] \right.$$
$$\left. + \sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} \frac{\partial^{|\beta|} u_h}{\partial x^\beta}(x) \frac{\partial^{|\alpha-\beta|+1} g_{ij}}{\partial x^{\alpha-\beta+(h)}}(x) \right\}$$

thus proving (i). Accordingly, we can define a unique map Φ^r by setting $\Phi^r(j_x^r g, j_x^{r+1} X) = \overline{X}^r_{j_x^r g}$. Since the map $X \mapsto \overline{X}^r$ is $\mathbb{R}$-linear, it is clear that Φ^r is a homomorphism of vector bundles as stated in (ii).

Let us define the subset $\mathcal{O}^r$ as follows. A point $j_x^r g$ belongs to $\mathcal{O}^r$ if, and only if, it has a neighbourhood $\mathcal{N}_{j_x^r g}$ such that the rank of $\Phi^r_{|\mathcal{N}_{j_x^r g}}$ is constant. From the very definition, $\mathcal{O}^r$ is an open subset and the rank of $\Phi^r_{|\mathcal{O}^r}$ is locally constant. Next, we prove that $\mathcal{O}^r$ is dense in $J^r(\mathcal{M})$. Let $\mathcal{U}$ be a non-empty open subset of $J^r(\mathcal{M})$. Since the rank of $\mathfrak{D}^r$ only takes a finite number of values, there exists a point $j_x^r g \in \mathcal{U}$ such that for every $j_{x'}^r g' \in \mathcal{U}$, $\operatorname{rk} \mathfrak{D}^r_{j_{x'}^r g'} \leqslant \operatorname{rk} \mathfrak{D}^r_{j_x^r g}$, and since the rank of a homomorphism of vector bundles is a lower semicontinuous function, $j_x^r g \in \mathcal{O}^r$. In order to prove that $\mathfrak{D}^r$ is involutive, we proceed as follows. Given a point $j_x^r g \in \mathcal{O}^r$, let $X_1, \ldots, X_k$ be vector fields on N such that $(\overline{X}_1)^r_{j_x^r g}, \ldots, (\overline{X}_k)^r_{j_x^r g}$ is a basis for $\mathfrak{D}^r_{j_x^r g}$. Then, there exists an open neighbourhood $\mathcal{N}_{j_x^r g}$ such that $\overline{X}^r_1, \ldots, \overline{X}^r_k$ is a basis of $\mathfrak{D}^r_{j_{x'}^r g'}$ for every $j_{x'}^r g' \in \mathcal{N}_{j_x^r g}$. Accordingly, any two vector fields ξ, ξ' belonging to $\mathfrak{D}^r_{|\mathcal{N}_{j_x^r g}}$ can be written as $\xi = \sum_i f_i(\overline{X}^r_i)$, $\xi' = \sum_j f'_j(\overline{X}^r_j)$, and, from formula (1), we obtain

$$[\xi, \xi'] = \sum_{i,j=1}^{k} \{ f_i \overline{X}^r_i(f'_j) \overline{X}^r_j - f'_j \overline{X}^r_j(f_i) \overline{X}^r_i + f_i f'_j \overline{[X_i, X_j]}^r \}$$

thus showing that $[\xi, \xi']$ also belongs to $\mathfrak{D}^r$, and complete the proof of (iii). Part (iv) follows directly from the definitions. $\square$

Corollary 11. On a neighbourhood of each point $j_x^r g \in \mathcal{O}^r$, the number of functionally independent metric differential invariants is $\dim J^r(\mathcal{M}) - \operatorname{rk} \Phi^r_{j_x^r g}$.

Proof. This follows from theorem 7 and the Frobenius theorem. $\square$

Our next goal is to determine the rank of Φ^r. In doing this we shall use normal coordinates which will always be assumed to be metric (i.e. associated with an orthonormal frame) and defined on a convex open neighbourhood of a given point $x \in N$ ([7], III section 8, IV section 3). The expansion of the metric in a normal coordinate system starts as follows (cf [9])

$$g_{ij} = g_{ij}(x) + \tfrac{1}{6} \sum_{k,l=1}^{n} (R_{ilkj}(x) + R_{jlki}(x)) x_k x_l + \cdots \tag{5}$$

where R_{ijkl} are the components of the curvature tensor, i.e.

$$R_{ijkl} = g(R(\partial/\partial x_k, \partial/\partial x_l)(\partial/\partial x_j), \partial/\partial x_i).$$

Taking derivatives in (5), we obtain

$$\frac{\partial g_{ij}}{\partial x_k}(x) = 0 \qquad 1 \leqslant i, j, k \leqslant n \tag{6}$$

and again taking derivatives

$$\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(x) = \tfrac{1}{3}(R_{iklj}(x) + R_{ilkj}(x)). \tag{7}$$

5. The rank of Φ^1

Theorem 12. With the same notation as in proposition 8, $\Phi^1(j_x^2 X) = \overline{X}^1_{j_x^1 g} = 0$ if, and only if, in a normal coordinate system $x_1, \ldots, x_n$ centred at $x \in N$, the following conditions hold true: for every $i, j, k = 1, \ldots, n$

$$u_i(x) = 0 \qquad g_{ii}(x)\frac{\partial u_i}{\partial x_j}(x) + g_{jj}(x)\frac{\partial u_j}{\partial x_i}(x) = 0 \qquad \frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = 0. \tag{8}$$

Hence, $\Phi^1 : J_x^2(TN) \to T_{j_x^1 g}(J^1\mathcal{M})$ is surjective at each point $j_x^1 g \in J^1(\mathcal{M})$.

Proof. From formula (4), we obtain

$$u_i(x) = 0 \qquad g_{ii}(x)\frac{\partial u_i}{\partial x_j}(x) + g_{jj}(x)\frac{\partial u_j}{\partial x_i}(x) = 0 \tag{9}$$

$$g_{jj}(x)\frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) + g_{ii}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) \\ + \sum_h \left(\frac{\partial u_h}{\partial x_i}(x)\frac{\partial g_{hj}}{\partial x_k}(x) + \frac{\partial u_h}{\partial x_j}(x)\frac{\partial g_{ih}}{\partial x_k}(x) + \frac{\partial u_h}{\partial x_k}(x)\frac{\partial g_{ij}}{\partial x_h}(x)\right) = 0. \tag{10}$$

By applying (6), equation (10) becomes

$$g_{jj}(x)\frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) + g_{ii}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = 0. \tag{11}$$

By permuting $(ik) \mapsto (ki)$ in (11), we have

$$g_{jj}(x)\frac{\partial^2 u_j}{\partial x_k \partial x_i}(x) + g_{kk}(x)\frac{\partial^2 u_k}{\partial x_j \partial x_i}(x) = 0. \tag{12}$$

Comparing (11) and (12), we obtain

$$g_{kk}(x)\frac{\partial^2 u_k}{\partial x_i \partial x_j}(x) = g_{ii}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x). \tag{13}$$

From (13) and (11), and again applying (13) after making the permutation $(ijk) \mapsto (jik)$, we obtain

$$\underline{g_{kk}(x)\frac{\partial^2 u_k}{\partial x_i \partial x_j}(x)} = g_{ii}(x)\frac{\partial^2 u_i}{\partial x_j \partial x_k}(x) = -g_{jj}(x)\frac{\partial^2 u_j}{\partial x_i \partial x_k}(x) = \underline{-g_{kk}(x)\frac{\partial^2 u_k}{\partial x_j \partial x_i}(x)}.$$

Accordingly,

$$\frac{\partial^2 u_k}{\partial x_i \partial x_j}(x) = 0$$

thus completing the proof of (8). Hence,

$$\operatorname{rk}\Phi^1 = \dim J_x^2(TN) - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}n(n^2+2n+3) = \dim T_{j_x^1 g}(J^1\mathcal{M}). \qquad \square$$

6. The rank of Φ^2

Lemma 13. With the same notation as in proposition 8 and theorem 12, $\Phi^2(j_x^3 X) = \overline{X}^2_{j_x^2 g} = 0$ if, and only if, in addition to equations (8), the following conditions hold true: for every $i, j, k, l = 1, \ldots, n$

$$\frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) = 0 \tag{14}$$

$$\sum_{h=1}^{n} \left(\frac{\partial u_h}{\partial x_i}(x) R_{hkjl}(x) + \frac{\partial u_h}{\partial x_j}(x) R_{hlik}(x) + \frac{\partial u_h}{\partial x_k}(x) R_{hilj}(x) + \frac{\partial u_h}{\partial x_l}(x) R_{hjki}(x) \right) = 0. \tag{15}$$

Proof. It follows from formula (3) that $\overline{X}^2_{j_x^2 g} = 0$ if, and only if, equations (8) hold and, furthermore, for every $i, j, k, l = 1, \ldots, n$

$$g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) + g_{jj}(x) \frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l}(x) + \lambda_{ijkl} = 0 \tag{16}$$

where we have set

$$\lambda_{ijkl} = \sum_{h=1}^{n} \left(\frac{\partial u_h}{\partial x_k}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_l}(x) + \frac{\partial u_h}{\partial x_l}(x) \frac{\partial^2 g_{ij}}{\partial x_h \partial x_k}(x) \right.$$
$$\left. + \frac{\partial u_h}{\partial x_i}(x) \frac{\partial^2 g_{hj}}{\partial x_k \partial x_l}(x) + \frac{\partial u_h}{\partial x_j}(x) \frac{\partial^2 g_{hi}}{\partial x_k \partial x_l}(x) \right).$$

Permuting the indices i, k in (16) and subtracting, we obtain

$$g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) - g_{kk}(x) \frac{\partial^3 u_k}{\partial x_j \partial x_i \partial x_l}(x) = \lambda_{kjil} - \lambda_{ijkl}$$

and permuting the indices j, k

$$g_{ii}(x) \frac{\partial^3 u_i}{\partial x_k \partial x_j \partial x_l}(x) - g_{jj}(x) \frac{\partial^3 u_j}{\partial x_k \partial x_i \partial x_l}(x) = \lambda_{jkil} - \lambda_{ikjl}. \tag{17}$$

By adding (16) and (17), we obtain

$$2 g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) = \lambda_{jkil} - \lambda_{ikjl} - \lambda_{ijkl}.$$

Formula (7) then yields

$$g_{ii}(x) \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}(x) = \frac{1}{3} \sum_{h=1}^{n} \left[\frac{\partial u_h}{\partial x_i}(x)(R_{klhj}(x) + R_{khlj}(x)) + \frac{\partial u_h}{\partial x_j}(x)(R_{ikhl}(x) + R_{ihkl}(x)) \right.$$
$$\left. + \frac{\partial u_h}{\partial x_k}(x)(R_{ihjl}(x) + R_{ijhl}(x)) + \frac{\partial u_h}{\partial x_l}(x)(R_{kihj}(x) + R_{khij}(x)) \right]. \tag{18}$$

Since the left-hand side of (18) is symmetric with respect to the indices j, k, l, permuting j and l and equating the corresponding right-hand sides, we obtain (15), thus also showing that the right-hand side of (18) vanishes and, hence, (18) reduces to (14). The proof is thus complete. □

Remark 14. Equation (15) is invariant under the group of order 8 generated by the permutations $\gamma : (ijkl) \mapsto (jkli)$ and $\tau : (ijkl) \mapsto (ilkj)$. Note that $\gamma \circ \tau = \tau \circ \gamma^2$. Accordingly, in examining (15), we only need to consider the following three cases: (i) $i \leqslant j \leqslant k \leqslant l$; (ii) $i \leqslant k < j \leqslant l$; and (iii) $i \leqslant j \leqslant l < k$.

Theorem 15.

(i) If $\dim N = n = 1$, for every $j_x^2 g \in J^2(\mathcal{M})$, $\Phi^2_{j_x^2 g}$ is bijective.

(ii) If $\dim N = n = 2$, for every $j_x^2 g \in J^2(\mathcal{M})$, $\operatorname{rk} \Phi^2_{j_x^2 g} = 19$.

(iii) For each $n \geqslant 3$, there exists a dense open subset $\mathcal{O}^{n,2} \subset J^2(\mathcal{M})$, such that for every $j_x^2 g \in \mathcal{O}^{n,2}$, $\Phi^2_{j_x^2 g}$ is injective.

Proof.

(i) From (8) and (14) it follows that Φ^2 is injective. Hence, $\operatorname{rk} \Phi^2_{j_x^2 g} = \dim J_x^3(TN) = 4 = \dim T_{j_x^2 g}(J^2 N)$.

(ii) Using the above remark, it is not difficult to check that equation (15) is identically satisfied if $n = 2$. Accordingly, from (8) and (14), it follows that a 3-jet $j_x^3 X \in \operatorname{Ker} \Phi^2_{j_x^2 g}$ is completely determined by $(\partial u_2/\partial x_1)(x)$. Hence, the kernel of Φ^2 is a vector subbundle of rank 1 and, therefore, $\operatorname{rk} \Phi^2_{j_x^2 g} = \dim J_x^3(TN) - 1 = 20 - 1 = 19$.

(iii) Let $r = \sum_{i,j} r_{ij}\, dx_i \otimes dx_j$ be the Ricci tensor of g; i.e. $r(X, Y) =$ trace of $Z \mapsto R(Z, X)(Y)$. Then we have $r_{ij}(x) = \sum_h g_{hh}(x) R_{hjhi}(x)$. Since r is symmetric, there exists a unique endomorphism $A : T_x(N) \to T_x N$, such that for every $X, Y \in T_x(N)$,

$$r(X, Y) = g(A(X), Y) = g(X, A(Y)). \tag{19}$$

Moreover, let B be the endomorphism given by $B = \sum_{i,j} (\partial u_i/\partial x_j)(x)\, d_x x_j \otimes (\partial/\partial x_i)_x$. From the second equation of (8), we deduce that for every $X, Y \in T_x(N)$,

$$g(B(X), Y) + g(X, B(Y)) = 0. \tag{20}$$

Let $\mathcal{O}^{n,2}$ be the set of points $j_x^2 g$ such that the eigenvalues of A in $T_x(N) \otimes \mathbb{C}$ are pairwise different. We shall prove that on $\mathcal{O}^{n,2}$, the unique solution of (15) is the trivial solution. Let us denote by $\langle X, Y\rangle$ the bilinear form induced by g_x on the complex vector space $T_x(N) \otimes \mathbb{C}$, so that (19) and (20) imply

$$\langle A(Z), W\rangle = \langle Z, A(W)\rangle \qquad \langle B(Z), W\rangle + \langle Z, B(W)\rangle = 0$$

for every $Z, W \in T_x(N) \otimes \mathbb{C}$. Letting $l = k$ in (15), multiplying by $g_{kk}(x)$ and using the second equation of (8), we obtain

$$\sum_{h=1}^{n} \left(r_{jh}(x) \frac{\partial u_h}{\partial x_i}(x) + r_{ih}(x) \frac{\partial u_h}{\partial x_j}(x) \right) = 0$$

or equivalently,

$$\langle AB(Z), W\rangle + \langle Z, AB(W)\rangle = 0$$

for every $Z, W \in T_x(N) \otimes \mathbb{C}$. Let $A(Z_i) = \lambda_i Z_i$ be the eigenvalues (and the eigenvectors) of A. From the above equations, we then obtain $\langle AB(Z_i), Z_j\rangle + \langle Z_i, AB(Z_j)\rangle = 0$ or equivalently, $\langle B(Z_i), A(Z_j)\rangle + \langle A(Z_i), B(Z_j)\rangle = 0$; i.e. $\lambda_j \langle B(Z_i), Z_j\rangle + \lambda_i \langle Z_i, B(Z_j)\rangle = 0$. Hence, $(\lambda_i - \lambda_j)\langle Z_i, B(Z_j)\rangle = 0$. By virtue of the hypothesis this implies $B(Z_j) = 0$, and since $Z_1, \ldots, Z_n$ is a basis of the complex tangent space, we have $B = 0$. □

7. The rank of Φ^r, $r \geqslant 3$

Theorem 16. For each $r \geqslant 3$, there exists a dense open subset $\mathcal{O}^{n,r} \subset J^r(\mathcal{M})$, such that for every $j^r_x g \in \mathcal{O}^{n,r}$, $\Phi^r_{j^r_x g}$ is injective.

Proof. First, we prove the theorem for $r = 3$. We distinguish two cases.

(i) $\dim N = n \neq 2$. For the sake of simplicity, we write $\mathcal{O}^{1,2} = J^2(\mathcal{M})$. From (i) and (iii) in theorem 15, we know that Φ^2 is injective on $\mathcal{O}^{n,2}$. We set $\mathcal{O}^{n,3} = p_{32}^{-1}(\mathcal{O}^{n,2})$. Assume $j^3_x g \in \mathcal{O}^{n,3}$. Since $\Phi^2_{|\mathcal{O}^{n,2}}$ is injective from formula (3), we have that: $j^4_x(X) \in \operatorname{Ker} \Phi^3_{j^3_x g}$ if, and only if, $j^3_x(X) = 0$ and, furthermore, for every $i, j, k, l, m = 1, \ldots, n$,

$$\frac{\partial^4 u_j}{\partial x_i \partial x_k \partial x_l \partial x_m}(x) g_{jj}(x) + \frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) g_{ii}(x) = 0. \tag{21}$$

Permuting i and k in (21) and subtracting, we have

$$g_{ii}(x)\frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) - g_{ii}(x)\frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) = 0$$

and permuting the indices j, k, and adding the equation thus obtained to (21), we have

$$\frac{\partial^4 u_i}{\partial x_j \partial x_k \partial x_l \partial x_m}(x) = 0 \qquad \text{for every } i, j, k, l, m = 1, \ldots, n.$$

Hence, $j^4_x(X) = 0$ and accordingly, $\Phi^3_{|\mathcal{O}^{n,3}}$ is injective.

(ii) $\dim N = n = 2$. From formula (3), we conclude that $j^4_x(X)$ belongs to the kernel of $\Phi^3_{j^3_x g}$, if and only if, in addition to equations (8) and (14), the following conditions hold true: for every $|\alpha| = 3, i, j = 1, 2$,

$$\sum_{h=1}^{2}\left(\frac{\partial u_h}{\partial x_i}(x)\frac{\partial^3 g_{hj}}{\partial x^\alpha}(x) + \frac{\partial u_h}{\partial x_j}(x)\frac{\partial^3 g_{hi}}{\partial x^\alpha}(x) + \sum_{k=1}^{2}\alpha_k \frac{\partial u_h}{\partial x_k}(x)\frac{\partial^3 g_{ij}}{\partial x^{\alpha-(k)+(h)}}(x)\right)$$
$$+ g_{jj}(x)\frac{\partial^4 u_j}{\partial x^{\alpha+(i)}}(x) + g_{ii}(x)\frac{\partial^4 u_i}{\partial x^{\alpha+(j)}}(x) = 0. \tag{22}$$

Recall that equation (15) is identically satisfied if $n = 2$. Moreover, the expansion given in (5) yields (cf [9])

$$\frac{\partial^3 g_{ij}}{\partial x_k \partial x_l \partial x_m}(x) = \frac{1}{6}\frac{\partial}{\partial x_k}(R_{imjl} + R_{iljm})(x) + \frac{1}{6}\frac{\partial}{\partial x_l}(R_{imjk} + R_{ikjm})(x)$$
$$+ \frac{1}{6}\frac{\partial}{\partial x_m}(R_{iljk} + R_{ikjl})(x).$$

From the above equation and (22), we then obtain

$$(\alpha = (3,0),\ i = j = 1)\text{:}\ \frac{\partial^4 u_1}{\partial x_1^4}(x) = 0$$

$$(\alpha = (3,0),\ i = 1, j = 2)\text{:}\ g_{22}(x)\frac{\partial^4 u_2}{\partial x_1^4}(x) + g_{11}(x)\frac{\partial^4 u_1}{\partial x_1^3 \partial x_2}(x) = 0$$

$$(\alpha=(3,0),\ i=j=2)\colon\ \frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_2}(x)+2g_{22}(x)\frac{\partial^4 u_2}{\partial x_1^3\partial x_2}(x)=0$$

$$(\alpha=(2,1),\ i=j=1)\colon\ \frac{\partial^4 u_1}{\partial x_1^3\partial x_2}(x)=0$$

$$(\alpha=(2,1),\ i=1,j=2)\colon\ -\frac{1}{3}\frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_2}(x)+g_{11}(x)\frac{\partial^4 u_1}{\partial x_1^2\partial x_2^2}(x)$$
$$+g_{22}(x)\frac{\partial^4 u_2}{\partial x_1^3\partial x_2}(x)=0$$

$$(\alpha=(2,1),\ i=j=2)\colon\ -\frac{1}{3}\frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_1}(x)+2g_{11}(x)\frac{\partial^4 u_2}{\partial x_1^2\partial x_2^2}(x)=0$$

$$(\alpha=(1,2),\ i=j=1)\colon\ \frac{1}{3}\frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_2}(x)+2g_{11}(x)\frac{\partial^4 u_1}{\partial x_1^2\partial x_2^2}(x)=0$$

$$(\alpha=(1,2),\ i=1,j=2)\colon\ \frac{1}{3}g_{22}(x)\frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_1}(x)+\frac{\partial^4 u_1}{\partial x_1\partial x_2^3}(x)$$
$$+g_{11}(x)g_{22}(x)\frac{\partial^4 u_2}{\partial x_1^2\partial x_2^2}(x)=0$$

$$(\alpha=(1,2),\ i=j=2)\colon\ \frac{\partial^4 u_2}{\partial x_1\partial x_2^3}(x)=0$$

$$(\alpha=(0,3),\ i=j=1)\colon\ -g_{22}(x)\frac{\partial u_2}{\partial x_1}\frac{\partial R_{1212}}{\partial x_1}(x)+2\frac{\partial^4 u_1}{\partial x_1\partial x_2^3}(x)=0$$

$$(\alpha=(0,3),\ i=1,j=2)\colon\ g_{22}(x)\frac{\partial^4 u_2}{\partial x_1\partial x_2^3}(x)+g_{11}(x)\frac{\partial^4 u_1}{\partial x_2^4}(x)=0$$

$$(\alpha=(0,3),\ i=j=2)\colon\ \frac{\partial^4 u_2}{\partial x_2^4}(x)=0.$$

It is not difficult to check that the above system is equivalent to saying that, for every $i,j,k,l,m=1,2$,

$$\frac{\partial^4 u_i}{\partial x_j\partial x_k\partial x_l\partial x_l}(x)=0$$

and

$$\frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_1}(x)=0 \qquad \frac{\partial u_2}{\partial x_1}(x)\frac{\partial R_{1212}}{\partial x_2}(x)=0.$$

Hence, in the dense open subset $\mathcal{O}^{2,3}$ of the 2-jets of metrics of a surface whose curvature satisfies $\|(\nabla R)(x)\|>0$, we have $j_x^4(X)=0$ and, therefore, $\Phi^3_{|\mathcal{O}^{2,3}}$ is injective.

By induction on r, we now prove the general statement of the theorem. For every $r\geqslant 3$, $n\geqslant 1$, we set $\mathcal{O}^{n,r}=p_{r3}^{-1}(\mathcal{O}^{n,3})$. Assume $j_x^{r+1}(X)\in\operatorname{Ker}\Phi^r_{j_x^r g}$, with $j_x^r g\in\mathcal{O}^{n,r}$.

Since $\overline{X}^r_{j^r_x g}$ projects onto $\overline{X}^{r-1}_{j^{r-1}_x g}$, it follows from the induction hypothesis that $j^r_x(X) = 0$ and, from formula (3), we thus deduce

$$\frac{\partial^{r+1} u_j}{\partial x^{\alpha+(i)}}(x) g_{jj}(x) + \frac{\partial^{r+1} u_i}{\partial x^{\alpha+(j)}}(x) g_{ii}(x) = 0$$

for every $i, j = 1, \ldots, n$, $|\alpha| = r$. Let k be an index such that $\alpha_k > 0$. We set $\beta = \alpha - (k)$, so that the above equation reads as follows

$$g_{ii}(x) \frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) + g_{jj}(x) \frac{\partial^{r+1} u_j}{\partial x_i \partial x_k \partial x^\beta}(x) = 0. \tag{23}$$

Permuting the indices i, k, in (23), and subtracting, we have

$$g_{ii}(x) \frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) - g_{kk}(x) \frac{\partial^{r+1} u_k}{\partial x_i \partial x_j \partial x^\beta}(x) = 0$$

and permuting the indices j, k and adding the above equation to (23), we obtain

$$\frac{\partial^{r+1} u_i}{\partial x_j \partial x_k \partial x^\beta}(x) = 0$$

thus proving that $j^{r+1}_x(X) = 0$ and completing the proof of the theorem. □

8. Calculating $i_{n,r}$

Theorem 17. On a dense open subset of $J^r(\mathcal{M})$, the number $i_{n,r}$ of functionally independent metric differential invariants is as follows.

(i) For every $n \geqslant 1$, $i_{n,0} = i_{n,1} = 0$.

(ii) For every $r \geqslant 0$, $i_{1,r} = 0$.

(iii) $i_{2,2} = 1$ and, for every $r \geqslant 3$, $i_{2,r} = \frac{1}{2}(r+1)(r-2)$.

(iv) For every $n \geqslant 3$, $r \geqslant 2$,

$$i_{n,r} = n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)} \binom{n+r}{r}.$$

Proof. First, we confine ourselves to the dense open subset $\mathcal{O}^r$ prescribed in theorem 10-(iii), where we know that the metric differential invariants of order r coincide with the ring of first integrals of the fundamental distribution.

(i) It follows from formula (2) that the vector fields $\overline{X}_{j^0_x g}$ span $T_{j^0_x g}(\mathcal{M})$; hence, $i_{n,0} = 0$. Since Φ^1 is surjective (theorem 12) from corollary 11, we conclude that $i_{n,1} = 0$.

(ii) From theorem 16, we know that Φ^r is injective (in fact $\mathcal{O}^{1,r} = J^r(\mathcal{M})$ in this case). From corollary 11, we thus have $i_{1,r} = \dim J^r(\mathcal{M}) - \operatorname{rk} \Phi^r_{j^r_x g} = \dim J^r(\mathcal{M}) - \dim J^{r+1}_x(TN) = (r+2) - (r+2) = 0$.

(iii) $i_{2,2} = 1$ follows directly from part (ii) of theorem 15. Moreover, the formula for $r \geqslant 3$ is a particular case of the formula in (iv).

(iv) From theorem 15(iii), theorem 16 and corollary 11, we have

$$\begin{aligned} i_{n,r} &= \dim J^r(\mathcal{M}) - \dim J_x^{r+1}(TN) \\ &= \left(n + \frac{n(n+1)}{2}\binom{n+r}{r}\right) - n\binom{n+r+1}{r+1} \\ &= n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)}\binom{n+r}{r}. \end{aligned}$$

□

Remark 18. For $n \geqslant 3$, there is a classical procedure in order to obtain second-order metric invariants, the so-called curvature invariants ([6], p 146). In the generic case, there is an essentially unique frame reducing g and its Ricci tensor to a canonical form. The invariants are the components of the Weyl tensor on that frame plus the n eigenvalues of the Ricci tensor. Let us calculate the dimension of the space of Weyl tensors. Following the same notation as [10], 1.105–116, we have that the space $\mathcal{C}E$ of curvature tensors (here, $E = T_x^*(N)$), breaks into three irreducible subspaces under the natural action of the orthogonal group $\mathcal{C}E = \mathcal{U}E \oplus \mathcal{Z}E \oplus \mathcal{W}E$, where $\dim \mathcal{U}E = 1$, $\dim \mathcal{Z}E = \frac{n(n+1)}{2} - 1$ (traceless symmetric 2-tensors) and $\mathcal{W}E$ are the Weyl tensors. Hence,

$$\begin{aligned} \dim \mathcal{W}E &= \dim \mathcal{C}E - \frac{n(n+1)}{2} \\ &= \dim S^2 \bigwedge^2 E - \dim \bigwedge^4 E - \frac{n(n+1)}{2} \\ &= \frac{n^4 - 7n^2 - 6n}{2}. \end{aligned}$$

Hence, the number of curvature invariants is

$$n + \frac{n^4 - 7n^2 - 6n}{2} = \frac{n^4 - 7n^2 + 6n}{2} = i_{n,2}.$$

Accordingly, this shows that the number of functionally independent curvature invariants is exactly the number of functionally independent second-order metric invariants.

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